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STRAINS AND JETS IN BLACK HOLE FIELDS

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Abstract. We study the behaviour of an initially spherical bunch of particles emitted along trajectories parallel to the symmetry axis of a Kerr black hole. We show that, under suitable conditions, curvature and inertial strains compete to generate jet-like structures.

1 Introduction

In recent papers, Bini, de Felice and Geralico (2006, 2007) considered how the relative deviations among the particles of a given bunch depend on the geometric properties of the frame adapted to a fiducial observer. Here we outline the main steps of their analysis which revealed how spacetime curvature and inertial strains compete to generate jet-like structures.

2 The relative deviation equation

Consider a bunch of test particles, i.e. a congruence $\mathcal{C}_U$ of timelike world lines with unit tangent vector U ($U \cdot U = -1$) parametrized by the proper time τ_U. In general, the lines of the congruence $\mathcal{C}_U$ are accelerated with acceleration $a(U) = \nabla_U U$. Let $\mathcal{C}_*$ be a fixed world line of the congruence which we consider as that of the "fiducial observer." The separation between the line $\mathcal{C}_*$ and a line of $\mathcal{C}_U$ is represented by a connecting vector Y, i.e. a vector undergoing Lie transport along U, namely $\pounds_U Y = 0$. The latter equation becomes:

$$\dot{Y} \equiv \frac{\mathrm{D}Y}{\mathrm{d}\tau_U} = \nabla_Y U = -(Y \cdot U)a(U) - K(U) \llcorner Y \ , \tag{2.1}$$

where the kinematical tensor $K(U)$ is defined as $K(U) = \omega(U) - \theta(U)$. Here the antisymmetric tensor $\omega(U)^\beta{}_\alpha$ represents the vorticity of the congruence $\mathcal{C}_U$

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and the symmetric tensor $\theta(U)^\beta{}_\alpha$ represents the expansion. The symbol $\llcorner$ stands for right-contraction operation among tensors. The covariant derivative along U of both sides of the Lie transport equation gives rise to the "relative deviation equation" which, once projected on a given spatial triad E_a, reads:

$$\ddot{Y}^a + \mathcal{K}_{(U,E)}{}^a{}_b Y^b = 0 \ , \tag{2.2}$$

where $\mathcal{K}_{(U,E)}{}^a{}_b = [T_{(\mathrm{fw},U,E)} - S(U) + \mathcal{E}(U)]^a{}_b$. Here $\mathcal{E}(U)$ is the electric part of the Riemann tensor relative to U namely $\mathcal{E}(U)^\alpha{}_\gamma = R^\alpha{}_{\beta\gamma\delta} U^\beta U^\delta$. The strain tensor S is defined as

$$S(U)_{ab} = \nabla(U)_b a(U)_a + a(U)_a a(U)_b \ ; \tag{2.3}$$

$S(U)$ depends only on the congruence $\mathcal{C}_U$ and not on the chosen spatial triad E_a, while the tensor T, given by:

$$\begin{aligned} T_{(\mathrm{fw},U,E)}{}^a{}_b &= \delta^a_b \omega^2_{(\mathrm{fw},U,E)} - \omega^a_{(\mathrm{fw},U,E)} \omega_{(\mathrm{fw},U,E)b} \\ &\quad - \epsilon^a{}_{bf} \dot{\omega}^f_{(\mathrm{fw},U,E)} - 2\epsilon^a{}_{fc} \omega^f_{(\mathrm{fw},U,E)} K(U)^c{}_b \ , \end{aligned} \tag{2.4}$$

as derived in detail in Bini et al. (2006), characterizes the given frame relative to a Fermi-Walker frame along the congruence. We like to stress here that the components of the deviation vector should be restricted to the reference observer's world line; they may behave differently according to the reference frame set up for their measurements.

3 Axial observers in Kerr spacetime

Consider Kerr spacetime, whose line element in Kerr-Schild coordinates $(t, x^1 = x, x^2 = y, x^3 = z)$ is given by

$$\mathrm{d}s^2 \quad = \quad g_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta \equiv (\eta_{\alpha\beta} + 2H k_\alpha k_\beta) \mathrm{d}x^\alpha \mathrm{d}x^\beta \ , \tag{3.1}$$

where $H = \mathcal{M} r^3/(r^4 + a^2 z^2)$, $\eta_{\alpha\beta}$ is the flat spacetime metric and

$$k_\alpha \mathrm{d}x^\alpha = -\mathrm{d}t - \frac{(rx + ay)\mathrm{d}x + (ry - ax)\mathrm{d}y}{r^2 + a^2} - \frac{z}{r}\mathrm{d}z \ , \tag{3.2}$$

with r implicitly defined by

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \ . \tag{3.3}$$

Here $\mathcal{M}$ and a are the total mass and specific angular momentum characterizing the spacetime. Consider now a set of particles moving along the z-direction, on and nearby the axis of symmetry, with four velocity

$$U = \gamma[m + \nu E(m)_{\hat{3}}] \ , \qquad \gamma = (1 - \nu^2)^{-1/2} \ , \tag{3.4}$$

where $m = \sqrt{(z^2 + a^2)/\Delta_z}\,\partial_t$, $E(m)_{\hat{3}} = \sqrt{\Delta_z/(z^2 + a^2)}[\partial_z + (2\mathcal{M}z/\Delta_z)\partial_t]$, $\Delta_z = z^2 - 2\mathcal{M}z + a^2$ and where the instantaneous linear velocity $\nu = \nu(z)$, relative to

the local static observers, is function of z. These particles are accelerated except perhaps those which move strictly on the axis of symmetry. Their history forms a congruence $\mathcal{C}_U$ of ∞^2 world lines and each of them can be parametrized by the pair (x, y) of the spatial coordinates. A frame adapted to this kind of orbits was found explitely in Bini at al. (2007) and termed $\{E(U)_{\hat{a}}\}$. The congruence is accelerated with non-vanishing components along the three spatial directions $E(U)_{\hat{a}}$. It is now possible to study the components of the vector Y connecting a specific curve of the congruence which we fix along the rotation axis with $x = 0 = y$ with tangent vector $\tilde{U}$ and nearby world lines of the same congruence. The reference world line is accelerated along the direction $E(\tilde{U})_{\hat{3}}$, i.e. $a(\tilde{U}) = a(\tilde{U})^{\hat{3}} E(\tilde{U})_{\hat{3}}$, with

$$a(\tilde{U})^{\hat{3}} \quad = \quad \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\gamma}{\sqrt{\tilde{g}_{33}}}\right) \ , \tag{3.5}$$

where $\tilde{g}_{33} = (z^2 + a^2)/\Delta_z$ denotes the restriction on the axis of that metric coefficient.

4 Tidal forces, frame induced deformations and strains

In order to determine the deviations measured by the "fiducial observer" $\tilde{U}$ with respect to the chosen frame we need to evaluate the kinematical fields of the congruence, namely the acceleration $a(U)$, the vorticity $\omega(U)$ and the expansion $\theta(U)$), the electric part of the Weyl tensor $\mathcal{E}(U)$, the strain tensor $S(U)$ and the characterization of the spatial triad $\{E(U)_{\hat{a}}\}$ with respect to a Fermi-Walker frame given by the tensor $T_{(\mathrm{fw},U,E)}$. All these quantities are calculated on the axis of symmetry so we can drop the tilde ($\tilde{}$); a lengthy calculation leads to the following results.

i) The only nonvanishing components of the kinematical tensors $\omega(U)$ and $\theta(U)$ are given by

$$\theta(U)_{\hat{3}\hat{3}} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\gamma\nu}{\sqrt{g_{33}}}\right) \ , \ \omega(U)_{\hat{1}\hat{2}} = \gamma(1+\nu)\frac{2a\mathcal{M}z}{\sqrt{\Delta_z}(z^2+a^2)^{3/2}} \ . \tag{4.1}$$

ii) The vorticity vector becomes $\omega_{(\mathrm{fw},U,E)} = \omega_{(\mathrm{fw},U,E)\hat{3}} E(U)_{\hat{3}}$ with $\omega_{(\mathrm{fw},U,E)\hat{3}} = \omega(U)_{\hat{1}\hat{2}}$.

iii) The electric part of the Weyl tensor is a diagonal matrix with respect to the adapted frame $\{E(U)_{\hat{a}}\}$, namely

$$\mathcal{E}(U) = \mathcal{M}z\frac{z^2 - 3a^2}{(z^2+a^2)^3}\,\mathrm{diag}\,[1, 1, -2] \ . \tag{4.2}$$

iv) The only nonvanishing components of the tensor $T_{(\mathrm{fw},U,E)}$ turn out to be

$$\begin{aligned} &T_{(\mathrm{fw},U,E)\hat{1}\hat{1}} = T_{(\mathrm{fw},U,E)\hat{2}\hat{2}} = -\omega_{(\mathrm{fw},U,E)}{}^2_{\hat{3}} \ , \\ &T_{(\mathrm{fw},U,E)\hat{1}\hat{2}} = -T_{(\mathrm{fw},U,E)\hat{2}\hat{1}} = -\dot{\omega}_{(\mathrm{fw},U,E)\hat{3}} \\ &= \frac{\gamma\nu}{\sqrt{g_{33}}}\frac{\mathrm{d}}{\mathrm{d}z}\omega_{(\mathrm{fw},U,E)\hat{3}} \ . \end{aligned} \tag{4.3}$$

v) The non-zero components of the strain tensor $S(U)^{\hat{a}}{}_{\hat{b}}$ can be written as

$$\begin{aligned}
S(U)_{\hat{1}\hat{1}} &= S(U)_{\hat{2}\hat{2}} = -\omega_{(\mathrm{fw},U,E)}{}^2_{\hat{3}} + \mathcal{M}z\frac{z^2-3a^2}{(z^2+a^2)^3}\ , \\
S(U)_{\hat{1}\hat{2}} &= -S(U)_{\hat{2}\hat{1}} = -\dot{\omega}_{(\mathrm{fw},U,E)\,\hat{3}}\ , \\
S(U)_{\hat{3}\hat{3}} &= \frac{\gamma}{\sqrt{g_{33}}}\frac{\mathrm{d}}{\mathrm{d}z}[a(U)^{\hat{3}}] + [a(U)^{\hat{3}}]^2\ .
\end{aligned} \tag{4.4}$$

From Eqs. (4.2)–(4.4) it follows that the deviation matrix $\mathcal{K}_{(U,E)}$ has only the non-zero component

$$\mathcal{K}_{(U,E)\,\hat{3}\hat{3}} = -\frac{\gamma}{\sqrt{g_{33}}}\frac{\mathrm{d}[a(U)^{\hat{3}}]}{\mathrm{d}z} - [a(U)^{\hat{3}}]^2 + \mathcal{E}(U)_{\hat{3}\hat{3}}\ ; \tag{4.5}$$

hence, as expected, the particles emitted nearby the axis in the z-direction will be relatively accelerated in the z-direction only. The spatial components of the connecting vector Y are then obtained by integrating the deviation equation (2.2) which now reads

$$\ddot{Y}^{\hat{1}} = 0\ , \quad \ddot{Y}^{\hat{2}} = 0\ , \quad \ddot{Y}^{\hat{3}} = -\mathcal{K}_{(U,E)\,\hat{3}\hat{3}}Y^{\hat{3}}\ . \tag{4.6}$$

Equations (4.6) can be analytically integrated to give

$$Y^{\hat{1}} = Y_0^{\hat{1}}, \qquad Y^{\hat{2}} = Y_0^{\hat{2}}, \qquad Y^{\hat{3}} = C\frac{\gamma\nu}{\sqrt{g_{33}}}\ , \tag{4.7}$$

where C is a constant. This result shows that the conditions imposed on the particles of the bunch to move parallel to the axis of rotation is assured by a suitable balancing among the gravitoelectric (curvature) tensor (4.2), the inertial tensor (4.3) and the strain tensor (4.4).

5 Reference world line uniformly accelerated

Let us now consider the case of the reference world line with tangent vector U constantly accelerated, namely with $a(U)^{\hat{3}} = A = \mathrm{const}$. The instantaneous linear velocity relative to a local static observer is given by

$$\nu_A = \left[1 - \frac{1}{g_{33}[\bar{\kappa} + A(z-z_0)]^2}\right]^{1/2}\ , \tag{5.1}$$

where the positive value has been selected for ν_A in order to consider outflows. $\bar{\kappa}$ is a constant which acquires the meaning of the conserved (Killing) energy of the particle if the latter is not accelerated. The solution of equation (4.7) is given by

$$Y^{\hat{1}} = Y_0^{\hat{1}}\ , \qquad Y^{\hat{2}} = Y_0^{\hat{2}}\ , \qquad Y^{\hat{3}} = Y_0^{\hat{3}}\frac{\gamma_A\nu_A}{\sqrt{g_{33}}\bar{\kappa}\nu_A^0}\ , \tag{5.2}$$

where $\nu_A^0 = \nu_A(z_0)$. Figure 1 shows the behaviour of an initially spherical bunch of particles in the $Y^{\hat{1}}$-$Y^{\hat{3}}$ plane for increasing values of the coordinate z. We clearly see a stretching along the z axis leading to a collimated axial outflow of matter, clearly suggestive of an "astrophysical jet." Figure 2 shows qualitatively the behaviour of an initially spherical bunch of particles. Note that in this case the acceleration acts contrarily to the curvature tidal effect; indeed, $S(U)$, $T_{(\mathrm{fw},U,E)}$ and $\mathcal{E}(U)$ act in competition leading to a quite unexpected result. It is easy to show that the stretching shown in Figs. 1 and 2 for uniformly accelerated outgoing particles persists also with a general acceleration. Moreover this behaviour does not depend of the acceleration mechanism itself. Observed jets (see Figs. 3 and 4 as an example) appear to contain spherical bunches of particles emerging from the central black hole.

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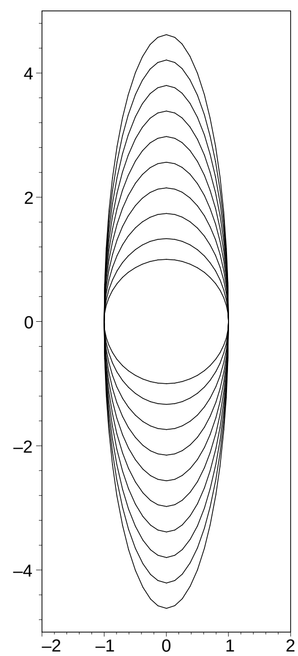

Fig. 1. An initially circular bunch of particles on the $y^{\hat{1}}$-$y^{\hat{3}}$ plane emitted along the z-axis at $z_0/\mathcal{M} = 2$ is shown to spread for the choice of parameters $a/\mathcal{M} = 0.5$, $\bar{\kappa} = 1.5$ and $\mathcal{M}\,A = 0.3$ as an example. The curves correspond to increasing values of the coordinate $z/\mathcal{M} = [2, 4, 6, 8, 10, 12, 14, 16, 18, 20]$.

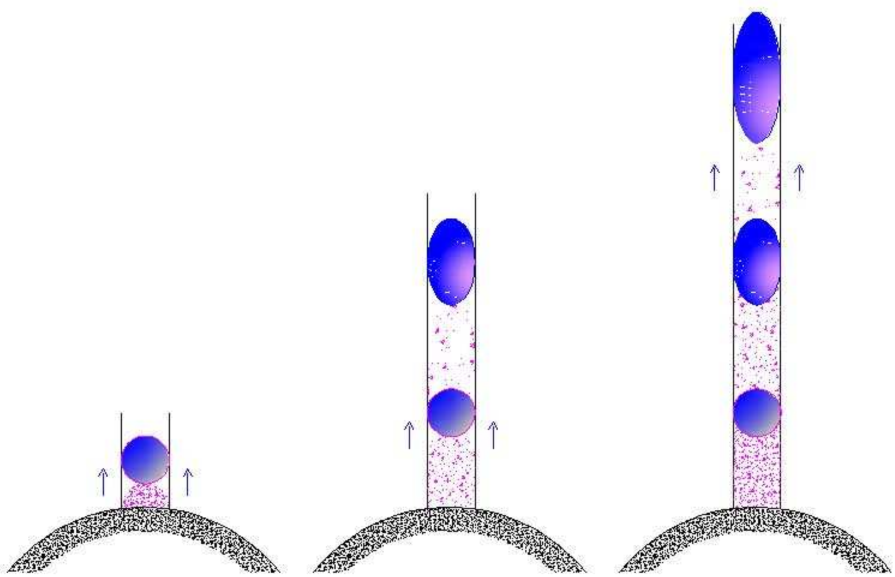

Fig. 2. The qualitative behaviour of an initially spherical bunch of particles moving out along the axis of a rotating black hole.

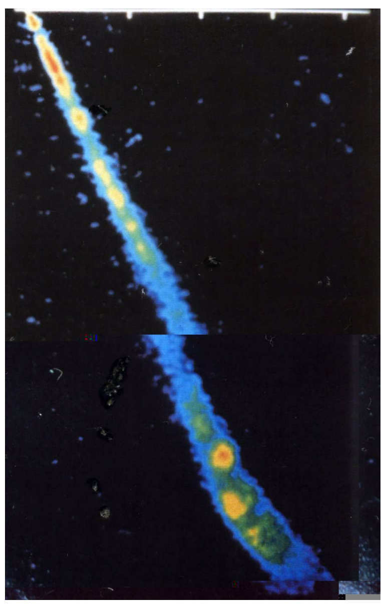

Fig. 3. The jet from the Galaxy M87 (Credit:J.A. Biretta et al. Hubble Heritage Team (STScl/AURA), NASA) We can see almost spherical bunches of particles along the jet.

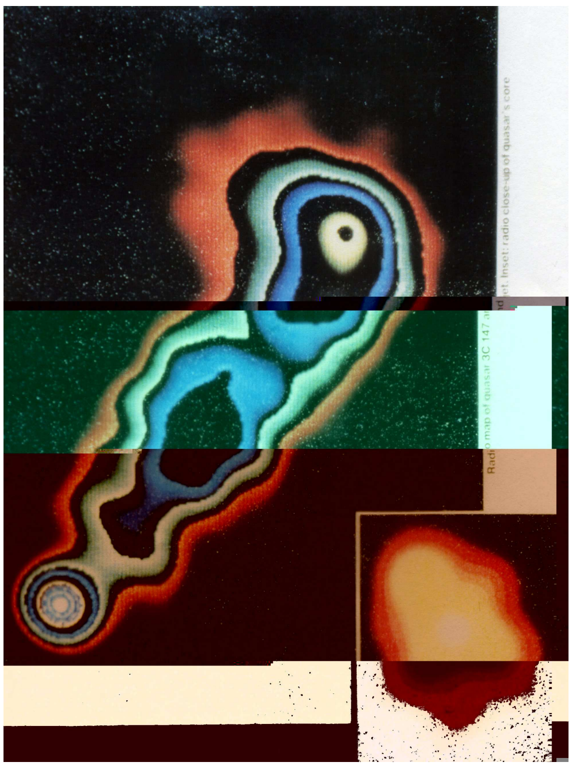

Fig. 4. Details of the jet from M87 close to the central black hole (Credit:J.A. Biretta et al. Hubble Heritage Team (STScl/AURA), NASA). Here we see a spherical bunch of particles apparently emerging from nearby the hole.